

## SUBDERIVATIVE-SUBDIFFERENTIAL DUALITY FORMULA

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**Abstract.** We provide a formula linking the radial subderivative to other subderivatives and subdifferentials for arbitrary extended real-valued lower semicontinuous functions.

**Keywords:** lower semicontinuity, radial subderivative, Dini subderivative, subdifferential.

**2010 Mathematics Subject Classification:** 49J52, 49K27, 26D10, 26B25.

## 1 Introduction

Tyrrell Rockafellar and Roger Wets [13, p. 298] discussing the duality between subderivatives and subdifferentials write

In the presence of regularity, the subgradients and subderivatives of a function  $f$  are completely dual to each other. [...] For functions  $f$  that aren't subdifferentially regular, subderivatives and subgradients can have distinct and independent roles, and some of the duality must be relinquished.

Jean-Paul Penot [12, p. 263], in the introduction to the chapter dealing with elementary and viscosity subdifferentials, writes:

In the present framework, in contrast to the convex objects, the passages from directional derivatives (and tangent cones) to subdifferentials (and normal cones, respectively) are one-way routes, because the first notions are nonconvex, while a dual object exhibits convexity properties.

In the chapter concerning Clarke subdifferentials [12, p. 357], he notes:

In fact, in this theory, a complete primal-dual picture is available: besides a normal cone concept, one has a notion of tangent cone to a set, and besides a subdifferential for a function one has a notion of directional derivative. Moreover, inherent convexity properties ensure a full duality between these notions. [...]. These facts represent great theoretical and practical advantages.

In this paper, we consider arbitrary extended real-valued lower semicontinuous functions and arbitrary subdifferentials. Yet we establish a duality formula linking subderivatives and subdifferentials of these functions. Moreover, we show that at points where the (lower semicontinuous) function satisfies a mild regularity property (called radial accessibility), the radial subderivative is always a lower bound for these dual expressions.

This lower bound is an equality in particular for convex functions, but also for various other classes of functions. For such functions, the radial subderivative can therefore be recovered from the subdifferential, and consequently the function itself, up to a constant, can be recovered from the subdifferential. This issue is discussed elsewhere.

## 2 Subderivatives

In the sequel,  $X$  is a real Banach space with unit ball  $B_X$ ,  $X^*$  is its topological dual, and  $\langle \cdot, \cdot \rangle$  is the duality pairing. For  $x, y \in X$ , we let  $[x, y] := \{x + t(y - x) : t \in [0, 1]\}$ ; the sets  $]x, y[$  and  $[x, y[$  are defined accordingly. Set-valued operators  $T : X \rightrightarrows X^*$  are identified with their graph  $T \subset X \times X^*$ . For a subset  $A \subset X$ ,  $x \in X$  and  $\lambda > 0$ , we let  $d_A(x) := \inf_{y \in A} \|x - y\|$  and  $B_\lambda(A) := \{y \in X : d_A(y) \leq \lambda\}$ . All extended-real-valued functions  $f : X \rightarrow ]-\infty, +\infty]$  are assumed to be lower semicontinuous (lsc) and *proper*, which means that the set  $\text{dom } f := \{x \in X : f(x) < \infty\}$  is non-empty.

For a lsc function  $f : X \rightarrow ]-\infty, +\infty]$ , a point  $\bar{x} \in \text{dom } f$  and a direction  $u \in X$ , we consider the following basic subderivatives (we essentially follow the terminology of Penot's textbook [12]):

- the (lower right Dini) *radial subderivative*:

$$f^r(\bar{x}; u) := \liminf_{t \searrow 0} \frac{f(\bar{x} + tu) - f(\bar{x})}{t},$$

and its upper strict version (the *Clarke subderivative*):

$$f^0(\bar{x}; u) := \limsup_{\substack{t \searrow 0 \\ (x, f(x)) \rightarrow (\bar{x}, f(\bar{x}))}} \frac{f(x + tu) - f(x)}{t};$$

- the (lower right Dini-Hadamard) *directional subderivative*:

$$f^d(\bar{x}; u) := \liminf_{\substack{t \searrow 0 \\ u' \rightarrow u}} \frac{f(\bar{x} + tu') - f(\bar{x})}{t},$$

and its upper strict version (the *Clarke-Rockafellar subderivative*):

$$f^\uparrow(\bar{x}; u) := \sup_{\delta > 0} \limsup_{\substack{t \searrow 0 \\ (x, f(x)) \rightarrow (\bar{x}, f(\bar{x}))}} \inf_{u' \in B_\delta(u)} \frac{f(x + tu') - f(x)}{t}.$$

It is immediate from these definitions that the following inequalities hold ( $\rightarrow$  means  $\leq$ ):

$$\begin{array}{ccc} f^r(\bar{x}; u) & \longrightarrow & f^0(\bar{x}; u) \\ \uparrow & & \uparrow \\ f^d(\bar{x}; u) & \longrightarrow & f^\uparrow(\bar{x}; u) \end{array}$$

It is well known (and easily seen) that for a function  $f$  locally Lipschitz at  $\bar{x}$ , we have  $f^r(\bar{x}; u) = f^d(\bar{x}; u)$  and  $f^0(\bar{x}; u) = f^\uparrow(\bar{x}; u)$ , whereas for a lsc convex  $f$ , we have  $f^d(\bar{x}; u) = f^\uparrow(\bar{x}; u)$ . A function  $f$  is declared to be *directionally regular* when such an equality holds. However, in general,  $f^d(\bar{x}; u) < f^\uparrow(\bar{x}; u)$ , and there are many other types of subderivatives  $f'$  which lie between  $f^d$  and  $f^\uparrow$ .

The inequality stated in the theorem below is (much) less elementary. It is the analytic form of Treiman's theorem [14] on the inclusion of the lower limit of Boulingand contingent cones at neighbouring points of  $\bar{x}$  into the Clarke tangent cone at  $\bar{x}$  in the context of

a Banach space (in finite dimensional spaces, equality holds between these objects, as was shown earlier by Cornet [3] and Penot [11]). A proof of this inequality (or equality in finite dimensional spaces) based on this geometrical approach was given by Ioffe [6] (see also Rockafellar-Wets [13, Theorem 8.18]). For a proof (in the general context of Banach spaces) using a multidirectional mean value inequality rather than the above geometric approach, see Correa-Gajardo-Thibault [4].

**Theorem 1** (Link between subderivatives). *Let  $X$  be a Banach space,  $f : X \rightarrow ]-\infty, +\infty]$  be lsc,  $\bar{x} \in \text{dom } f$  and  $u \in X$ . Then:*

$$f^\uparrow(\bar{x}; u) \leq \sup_{\varepsilon > 0} \limsup_{x \rightarrow \bar{x}} \inf_{u' \in B(u, \varepsilon)} f^d(x; u').$$

### 3 Subdifferentials

Given a lsc function  $f : X \rightarrow ]-\infty, +\infty]$  and a point  $\bar{x} \in \text{dom } f$ , we consider the following two basic subsets of the dual space  $X^*$ :

- the *Moreau-Rockafellar subdifferential* (the subdifferential of convex analysis):

$$\partial_{MR}f(\bar{x}) := \{x^* \in X^* : \langle x^*, y - \bar{x} \rangle + f(\bar{x}) \leq f(y), \forall y \in X\};$$

- the *Clarke subdifferential*, associated to the Clarke-Rockafellar subderivative:

$$\partial_Cf(\bar{x}) := \{x^* \in X^* : \langle x^*, u \rangle \leq f^\uparrow(\bar{x}; u), \forall u \in X\}.$$

All the classical subdifferentials (proximal, Fréchet, Hadamard, Ioffe, Michel-Penot, ...) lie between these two subsets. It is well known that for a lsc convex  $f$ ,  $\partial_{MR}f = \partial_Cf$ , so all the classical subdifferentials coincide with  $\partial_{MR}f$ .

In the sequel, we call *subdifferential* any operator  $\partial$  that associates a set-valued mapping  $\partial f : X \rightrightarrows X^*$  to each function  $f$  on  $X$  so that

$$\partial_{MR}f \subset \partial f \subset \partial_Cf$$

and the following *Separation Principle* is satisfied in  $X$ :

(SP) *For any lsc  $f, \varphi$  with  $\varphi$  convex Lipschitz near  $\bar{x} \in \text{dom } f$ , if  $f + \varphi$  admits a local minimum at  $\bar{x}$ , then  $0 \in \widehat{\partial}f(\bar{x}) + \partial\varphi(\bar{x})$ , where*

$$\widehat{\partial}f(\bar{x}) := \{\bar{x}^* \in X^* : \text{there is a net } ((x_\nu, x_\nu^*))_\nu \subset \partial f \text{ with}$$

$$(x_\nu, f(x_\nu)) \rightarrow (\bar{x}, f(\bar{x})), x_\nu^* \xrightarrow{w^*} \bar{x}^*, \limsup_\nu \langle x_\nu^*, x_\nu - \bar{x} \rangle \leq 0\}. \quad (1)$$

*Remark 1.1.* (a) In our paper [9], the set  $\widehat{\partial}f(\bar{x})$  defined in (1) is called the *weak\*-controlled closure* of the set-valued map  $\partial f$  at point  $\bar{x}$ . The reason to consider such a closure is that, even for a convex lsc function  $f$ , the a priori simpler *strong*  $\times$  *weak\**-closure of the graph of  $\partial f = \partial_{MR}f$  is too big for the Separation Principle to be meaningful. The graph of  $\partial_{MR}f$  is not *strong*  $\times$  *weak\**-closed in general: see, e.g., [8] for a discussion on what would be sufficient to add to the *strong*  $\times$  *weak\** topology on  $X \times X^*$  to guarantee the closure of such graphs. More precisely, the graph of the convex subdifferential is *strong*  $\times$  *weak\**-closed for each lsc convex function if and only if  $X$  is finite dimensional (see [1]). It is worth noting (and easily seen) that, as expected, always  $\partial_{MR}f = \widehat{\partial}_{MR}f$ .

(b) If we require the net  $((x_\nu, x_\nu^*))_\nu \subset \partial f$  in (1) to be actually a sequence  $((x_n, x_n^*))_n$ ,  $n \in \mathbb{N}$  (in which case the control assertion  $\limsup_n \langle x_n^*, x_n - \bar{x} \rangle \leq 0$  is automatically satisfied), we obtain the so-called ‘limiting subdifferentials’. A widely used such limiting subdifferential is the weak\* sequential closure of the Fréchet subdifferential, known as the *Mordukhovich subdifferential*.

(c) The Separation Principle (SP) is a very simple property expected to be satisfied by a subdifferential  $\partial$  in a Banach space  $X$ . This property is actually equivalent to various other properties of the subdifferential  $\partial$  in the Banach space  $X$ : see [9].

We recall that the Clarke subdifferential, the Michel-Penot subdifferential and the Ioffe subdifferential satisfy the Separation Principle in any Banach space. The elementary subdifferentials (proximal, Fréchet, Hadamard,  $\dots$ ), as well as their viscosity and limiting versions, satisfy the Separation Principle in appropriate Banach spaces: the Fréchet subdifferential in Asplund spaces, the Hadamard subdifferential in separable spaces, the proximal subdifferential in Hilbert spaces. The Moreau-Rockafellar subdifferential does not satisfy the Separation Principle for the whole class of lsc (non necessarily convex) functions: it is not a subdifferential for this class. See, e.g. [7, 9, 12] and the references therein.

The following link between the radial subderivative and arbitrary subdifferentials was established in [10, Theorem 2.1] (see also [9, Theorem 3.2]):

**Theorem 2** (Link between radial subderivative and subdifferentials). *Let  $X$  be a Banach space,  $f : X \rightarrow ]-\infty, +\infty]$  be lsc,  $\bar{x} \in \text{dom } f$  and  $u \in X$ . Then, there is a sequence  $((x_n, x_n^*)) \subset \partial f$  such that  $x_n \rightarrow \bar{x}$ ,  $f(x_n) \rightarrow f(\bar{x})$ ,*

$$f^r(\bar{x}; u) \leq \liminf_n \langle x_n^*, u \rangle \text{ and } \limsup_n \langle x_n^*, x_n - \bar{x} \rangle \leq 0.$$

## 4 Subderivative-subdifferential duality formula

A sequence  $(x_n) \subset X$  is said to be *directionally convergent* to  $\bar{x}$  in the direction  $v \in X$ , written  $x_n \rightarrow_v \bar{x}$ , if there are two sequences  $t_n \searrow 0$  (that is,  $t_n \rightarrow 0$  with  $t_n > 0$ ) and  $v_n \rightarrow v$  such that  $x_n = \bar{x} + t_n v_n$  for all  $n$ ; equivalently: for every  $\varepsilon > 0$  the sequence  $(x_n - \bar{x})$  eventually lies in the open drop  $]0, \varepsilon B(v, \varepsilon)[ := \{tv' : 0 < t < \varepsilon, v' \in B(v, \varepsilon)\}$ . Observe that for  $v = 0$ ,  $]0, \varepsilon B(v, \varepsilon)[ = B(0, \varepsilon^2)$  so  $x_n \rightarrow_v \bar{x}$  simply means  $x_n \rightarrow \bar{x}$ . We let  $D(\bar{x}, v, \varepsilon) := \bar{x} + ]0, \varepsilon B(v, \varepsilon)[$ .

The *support function* of a set  $S \subset X^*$  is the function  $\sup\langle S, \cdot \rangle : X \rightarrow \overline{\mathbb{R}}$  given by

$$u \in X \mapsto \sup\langle S, u \rangle := \sup\{\langle x^*, u \rangle : x^* \in S\}.$$

By analogy with subdifferentials, in the theorem below we call *subderivative* of a function  $f : X \rightarrow ]-\infty, +\infty]$  any function  $f' : \text{dom } f \times X \rightarrow \overline{\mathbb{R}}$  lying between  $f^d$  and  $f^\uparrow$ , that is:

$$f^d \leq f' \leq f^\uparrow.$$

Combining Theorem 1 and Theorem 2 we obtain a general duality formula relating subderivatives and subdifferentials:

**Theorem 3** (Subderivative-subdifferential duality formula). *Let  $X$  be a Banach space,  $f : X \rightarrow ]-\infty, +\infty]$  be lsc,  $\bar{x} \in \text{dom } f$  and  $u \in X$ . Then, for any direction  $v \in X$  and any real number  $\alpha \geq 0$ , one has*

$$\limsup_{x \rightarrow_v \bar{x}} f^r(x; u + \alpha(\bar{x} - x)) = \limsup_{x \rightarrow_v \bar{x}} f'(x; u + \alpha(\bar{x} - x)) \quad (2a)$$

$$= \limsup_{x \rightarrow_v \bar{x}} \langle \partial f(x), u + \alpha(\bar{x} - x) \rangle. \quad (2b)$$

*Proof. First step.* We claim that

$$\limsup_{x \rightarrow_v \bar{x}} f^\uparrow(x; u + \alpha(\bar{x} - x)) \leq \limsup_{x \rightarrow_v \bar{x}} f^d(x; u + \alpha(\bar{x} - x)). \quad (3)$$

To prove this inequality, we take  $\lambda \in \mathbb{R}$  such that

$$\limsup_{x \rightarrow_v \bar{x}} f^d(x; u + \alpha(\bar{x} - x)) < \lambda \quad (4)$$

and show that  $\lambda$  is greater than or equal to the left-hand side of (3).

From (4) we can find  $\delta > 0$  such that

$$x \in D(\bar{x}, v, \delta) \Rightarrow f^d(x; u + \alpha(\bar{x} - x)) < \lambda. \quad (5)$$

Let  $z = \bar{x} + tv' \in D(\bar{x}, v, \delta/2)$  and let  $\mu < f^\uparrow(z; u + \alpha(\bar{x} - z))$ . By Theorem 1, there exist  $\varepsilon > 0$  and  $x \in B(z, \rho)$ , with  $0 < \rho \leq t\delta/2$  and  $\alpha\rho \leq \varepsilon$ , such that

$$\mu < f^d(x; u + \alpha(\bar{x} - z) + w) \text{ for every } w \in B(0, \varepsilon). \quad (6)$$

Since  $\alpha\|z - x\| \leq \alpha\rho \leq \varepsilon$ , putting  $w = \alpha(z - x)$  in (6), we infer that

$$\mu < f^d(x; u + \alpha(\bar{x} - x)). \quad (7)$$

Since  $\|x - \bar{x} - tv'\| = \|x - z\| \leq \rho \leq t\delta/2$ , we have  $v'' := (x - \bar{x})/t \in B(v', \delta/2) \subset B(v, \delta)$ , showing that  $x = \bar{x} + tv'' \in D(\bar{x}, v, \delta)$ . Therefore, by (5),

$$f^d(x; u + \alpha(\bar{x} - x)) < \lambda. \quad (8)$$

Combining (7) and (8), we derive that  $\mu < \lambda$ . Since  $\mu$  was arbitrarily chosen less than  $f^\uparrow(z; u + \alpha(\bar{x} - z))$ , we conclude that

$$z \in D(\bar{x}, v, \delta/2) \Rightarrow f^\uparrow(z; u + \alpha(\bar{x} - z)) < \lambda,$$

hence,

$$\limsup_{x \rightarrow_v \bar{x}} f^\uparrow(x; u + \alpha(\bar{x} - x)) \leq \lambda.$$

This completes the proof of (3).

*Second step.* We claim that

$$\limsup_{x \rightarrow_v \bar{x}} f^r(x; u + \alpha(\bar{x} - x)) \leq \limsup_{x \rightarrow_v \bar{x}} \sup \langle \partial f(x), u + \alpha(\bar{x} - x) \rangle. \quad (9)$$

As in the first step, to prove this inequality we take  $\lambda \in \mathbb{R}$  such that

$$\limsup_{x \rightarrow_v \bar{x}} \sup \langle \partial f(x), u + \alpha(\bar{x} - x) \rangle < \lambda \quad (10)$$

and show that  $\lambda$  is greater than or equal to the left-hand side of (9).

From (10) we can find  $\delta > 0$  such that

$$x \in D(\bar{x}, v, \delta) \Rightarrow \sup \langle \partial f(x), u + \alpha(\bar{x} - x) \rangle < \lambda. \quad (11)$$

Let  $z = \bar{x} + tv' \in D(\bar{x}, v, \delta/2)$ . By Theorem 2, for any  $\mu < f^r(z; u + \alpha(\bar{x} - z))$  and  $\varepsilon > 0$  there exist  $x \in B(z, t\delta/2)$  and  $x^* \in \partial f(x)$  such that

$$\mu < \langle x^*, u + \alpha(\bar{x} - z) \rangle \text{ and } \langle x^*, x - z \rangle \leq \varepsilon.$$

As above, we can verify that  $x \in D(\bar{x}, v, \delta)$ . Therefore, by (11),

$$\mu < \langle x^*, u + \alpha(\bar{x} - z) \rangle = \langle x^*, u + \alpha(\bar{x} - x) \rangle + \alpha \langle x^*, x - z \rangle < \lambda + \alpha\varepsilon.$$

Since  $\mu$  and  $\varepsilon$  were arbitrary, we derive that

$$z \in D(\bar{x}, v, \delta/2) \Rightarrow f^r(z; u + \alpha(\bar{x} - z)) < \lambda,$$

showing that (9) holds.

*Third step.* Since  $\partial f \subset \partial_C f$ , we have  $\sup \langle \partial f(z), u' \rangle \leq \sup \langle \partial_C f(z), u' \rangle \leq f^\uparrow(z; u')$  for every  $u' \in X$ . Hence, the right-hand side of (9) is less than or equal to the left-hand side of (3). On the other hand,  $f^d \leq f^r$ . So all the expressions in formulas (3) and (9) are equal. The desired set of equalities (2a)–(2b) follows because  $f^d \leq f' \leq f^\uparrow$ .  $\square$

*Remark 3.1.* (a) In the special case  $v = 0$  and  $\alpha = 0$ , the formula (2a) was proved by Borwein-Strójas [2, Theorem 2.1 and Corollary 2.3]

(b) For  $f$  locally Lipschitz at  $\bar{x}$ , the formulas (2) do not depend on  $\alpha \geq 0$  since

$$\limsup_{x \rightarrow_v \bar{x}} f^d(x; u + \alpha(\bar{x} - x)) = \limsup_{x \rightarrow_v \bar{x}} f^d(x; u).$$

But they depend on the direction  $v \in X$ : for  $f : x \in \mathbb{R} \mapsto f(x) := -|x|$  and  $u \neq 0$ , one has

$$\limsup_{x \rightarrow_u 0} f'(x; u) = -|u| < \limsup_{x \rightarrow 0} f'(x; u) = |u|.$$

(c) For arbitrary lsc  $f$ , the value of the expressions in (2) depends on  $\alpha \geq 0$  even for convex  $f$ . Indeed, as was recalled in Remark 1.1 (a), the graph of the subdifferential

$$\partial_{MR} f(\bar{x}) = \{x^* \in X^* : \langle x^*, u \rangle \leq f^r(\bar{x}; u), \forall u \in X\}$$

is generally not *strong*  $\times$  *weak*<sup>\*</sup>-closed. Therefore, for an arbitrary lsc convex  $f$  the function  $x \mapsto f^r(x; u)$  is generally not upper semicontinuous, that is

$$f^r(\bar{x}; u) < \limsup_{x \rightarrow \bar{x}} f^r(x; u),$$

while always (see Proposition 4 below)

$$f^r(\bar{x}; u) = \inf_{\alpha \geq 0} \limsup_{x \rightarrow \bar{x}} f^r(x; u + \alpha(\bar{x} - x)).$$

**Proposition 4** (Radial subderivative for convex lsc functions). *Let  $X$  be a Banach space,  $f : X \rightarrow ]-\infty, +\infty]$  be convex lsc,  $\bar{x} \in \text{dom } f$  and  $u \in X$ . Then,*

$$f^r(\bar{x}; u) = \inf_{\alpha \geq 0} \limsup_{x \rightarrow \bar{x}} f^r(x; u + \alpha(\bar{x} - x)). \quad (12)$$

*Proof.* Of course,  $f^r(\bar{x}; u)$  is always not greater than the expression of the right-hand side of (12). It is not less either since, for every  $t > 0$ ,

$$\limsup_{x \rightarrow \bar{x}} f^r(x; tu + \bar{x} - x) \leq \limsup_{x \rightarrow \bar{x}} (f(\bar{x} + tu) - f(x)) = f(\bar{x} + tu) - f(\bar{x}),$$

hence, writing  $\alpha = 1/t$  for  $t > 0$ ,

$$\begin{aligned} \inf_{\alpha \geq 0} \limsup_{x \rightarrow \bar{x}} f^r(x; u + \alpha(\bar{x} - x)) &\leq \inf_{t > 0} \limsup_{x \rightarrow \bar{x}} \frac{1}{t} f^r(x; tu + \bar{x} - x) \\ &\leq \inf_{t > 0} \frac{f(\bar{x} + tu) - f(\bar{x})}{t} = f^r(\bar{x}; u). \end{aligned} \quad \square$$

## 5 Radially accessible functions

A lsc function  $f : X \rightarrow ]-\infty, +\infty]$  is said to be *radially accessible* at  $\bar{x} \in \text{dom } f$  from a direction  $u \in X$  provided

$$f(\bar{x}) = \liminf_{t \searrow 0} f(\bar{x} + tu),$$

or equivalently, provided there exists a sequence  $t_n \searrow 0$  such that  $f(\bar{x} + t_n u) \rightarrow f(\bar{x})$ . (The case  $u = 0$  is a tautology.)

*Examples.* 1. Every lsc function  $f : X \rightarrow ]-\infty, +\infty]$  which is radially upper semicontinuous at  $\bar{x} \in \text{dom } f$  from  $u$  is evidently radially accessible at  $\bar{x}$  from  $u$ . This is the case of convex lsc functions  $f$  for any  $u \in X$  such that  $\bar{x} + u \in \text{dom } f$ .

2. If  $f^r(\bar{x}; u) < \infty$ , then  $f$  is radially accessible at  $\bar{x}$  from  $u$ . Indeed, let  $\gamma \in \mathbb{R}$  such that  $f^r(\bar{x}; u) < \gamma$ . Then, there exists  $t_n \searrow 0$  such that  $f(\bar{x} + t_n u) \leq f(\bar{x}) + \gamma t_n$ , and consequently  $\limsup_n f(\bar{x} + t_n u) \leq f(\bar{x})$ . The condition  $f^r(\bar{x}; u) < \infty$  however is not necessary: the continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) := \sqrt{|x|}$  has  $f^r(0; u) = \infty$  for any  $u \neq 0$ .

3. The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$f(x) := \begin{cases} 0 & \text{if } x = 0 \text{ or } x = 1/n, \text{ for } n = 1, 2, \dots \\ 1 & \text{otherwise.} \end{cases}$$

is lsc on  $\mathbb{R}$ , not upper semicontinuous at 0 along the ray  $\mathbb{R}_+ u$  for  $u > 0$  but radially accessible at 0 from such  $u > 0$ .

4. The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$f(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

is lsc on  $\mathbb{R}$  but not radially accessible at 0 from  $u = 1$ . We notice that  $f^r(0; 1) = +\infty$ , while  $f^r(x; 1) = 0$  for any  $x > 0$ .

Radial accessibility is a mild regularity property. Yet this property leads to a more consistent behaviour of subdifferentials and subderivatives. We give two illustrations. Assume the lsc function  $f$  is radially accessible at  $\bar{x} \in \text{dom } f$  from a direction  $u$ . Then, there is a sequence  $(x_n)$  graphically and *directionally* converging to  $\bar{x}$  from  $u$  such that the subdifferential  $\partial f(x_n)$  is non-empty (Theorem 5), and there is a sequence  $(x'_n)$  graphically and *radially* converging to  $\bar{x}$  from  $u$  such that the lower limit of the radial subderivatives  $f'(x'_n; u)$  dominates the value  $f'(\bar{x}; u)$  (Theorem 7).

We recall the statement of Ekeland's variational principle [5]:

*Variational Principle.* For any lsc function  $f$  defined on a closed subset  $S$  of a Banach space,  $\bar{x} \in \text{dom } f$  and  $\varepsilon > 0$  such that  $f(\bar{x}) \leq \inf f(S) + \varepsilon$ , and for any  $\lambda > 0$ , there exists  $x_\lambda \in S$  such that  $\|x_\lambda - \bar{x}\| \leq \lambda$ ,  $f(x_\lambda) \leq f(\bar{x})$  and the function  $x \mapsto f(x) + (\varepsilon/\lambda)\|x - x_\lambda\|$  attains its minimum on  $S$  at  $x_\lambda$ .

**Theorem 5** (Directional density of subdifferentials). *Let  $X$  be a Banach space,  $f : X \rightarrow ]-\infty, +\infty]$  be lsc,  $\bar{x} \in \text{dom } f$  and  $u \in X$  such that  $f$  is radially accessible at  $\bar{x}$  from  $u$ . Then, there exists a sequence  $((x_n, x_n^*))_n \subset \partial f$  such that  $x_n \rightarrow_u \bar{x}$ ,  $f(x_n) \rightarrow f(\bar{x})$  and  $\limsup_n \langle x_n^*, x_n - \bar{x} \rangle \leq 0$ .*

*Proof.* Let  $\varepsilon > 0$ . Since  $f$  is lsc at  $\bar{x}$ , there exists  $\delta \in ]0, \varepsilon^2[$  such that

$$f(\bar{x}) \leq \inf f(B_\delta(\bar{x})) + \varepsilon^2, \quad (13)$$

and since  $f(\bar{x}) = \liminf_{t \searrow 0} f(\bar{x} + tu)$ , there exists  $\mu > 0$  such that  $\mu(\varepsilon + \|u\|) < \delta$  and  $f(\bar{x} + \mu u) \leq f(\bar{x}) + \varepsilon^2$ . Summarizing, we can find real numbers  $\delta$  and  $\mu$  satisfying

$$0 < \mu(\varepsilon + \|u\|) < \delta < \varepsilon^2, \text{ and} \quad (14a)$$

$$f(\bar{x} + \mu u) \leq \inf f(B_\delta(\bar{x})) + \varepsilon^2. \quad (14b)$$

Now we apply Ekeland's variational principle to  $f$  on the set  $B_\delta(\bar{x})$  at point  $\bar{x} + \mu u$  with  $\lambda = \mu\varepsilon$ . Observe that the ball  $B_\lambda(\bar{x} + \mu u)$  is contained in the ball  $B_\delta(\bar{x})$  by (14a). We therefore obtain a point  $x_\varepsilon \in X$  such that

$$\|x_\varepsilon - (\bar{x} + \mu u)\| < \mu\varepsilon, \quad f(x_\varepsilon) \leq f(\bar{x} + \mu u), \text{ and} \quad (15a)$$

$$y \mapsto f(y) + (\varepsilon/\mu)\|y - x_\varepsilon\| \text{ admits a local minimum at } x_\varepsilon. \quad (15b)$$

In view of (15b), we may apply the Separation Principle at point  $x_\varepsilon$  with the convex Lipschitz function  $\varphi : y \mapsto (\varepsilon/\mu)\|y - x_\varepsilon\|$  to obtain a subgradient  $x_\varepsilon^* \in \widehat{\partial} f(x_\varepsilon)$  such that

$$\|x_\varepsilon^*\| \leq \varepsilon/\mu. \quad (16)$$



Now, take  $(\bar{x}_\varepsilon, \bar{x}_\varepsilon^*) \in \partial f$  such that

$$\|\bar{x}_\varepsilon - x_\varepsilon\| < \mu\varepsilon, \quad |f(\bar{x}_\varepsilon) - f(x_\varepsilon)| < \varepsilon^2, \quad \text{and} \quad (17a)$$

$$\langle \bar{x}_\varepsilon^* - x_\varepsilon^*, \bar{x} - x_\varepsilon \rangle > -\varepsilon, \quad \langle \bar{x}_\varepsilon^*, \bar{x}_\varepsilon - x_\varepsilon \rangle < \varepsilon. \quad (17b)$$

It follows from the first parts of (15a) and (17a) that

$$\|\bar{x}_\varepsilon - (\bar{x} + \mu u)\| < 2\mu\varepsilon, \quad (18)$$

from the second parts of (15a) and (17a) combined with (13) and (14b) that

$$|f(\bar{x}_\varepsilon) - f(\bar{x})| \leq 2\varepsilon^2, \quad (19)$$

and from (16) and (17b) that

$$\langle \bar{x}_\varepsilon^*, \bar{x} - \bar{x}_\varepsilon \rangle > -\varepsilon(\varepsilon + \|u\| + 2), \quad (20)$$

since

$$\begin{aligned} \langle \bar{x}_\varepsilon^*, \bar{x} - \bar{x}_\varepsilon \rangle &= \langle x_\varepsilon^*, \bar{x} - x_\varepsilon \rangle + \langle \bar{x}_\varepsilon^* - x_\varepsilon^*, \bar{x} - x_\varepsilon \rangle + \langle \bar{x}_\varepsilon^*, x_\varepsilon - \bar{x}_\varepsilon \rangle \\ &> -\|x_\varepsilon^*\| \|x_\varepsilon - \bar{x}\| - 2\varepsilon \\ &> -(\varepsilon/\mu)\mu\varepsilon + \mu\|u\| - 2\varepsilon = -\varepsilon(\varepsilon + \|u\| + 2). \end{aligned}$$

Therefore, if for every  $n \in \mathbb{N}$ , we let  $\varepsilon = 1/n$  and choose  $\mu_n = \mu$  satisfying (14a), so that  $0 < \mu_n < 1/n$ , we obtain a sequence  $((x_n, x_n^*))_n$  in  $\partial f$  by setting  $x_n := \bar{x}_\varepsilon$  and  $x_n^* := \bar{x}_\varepsilon^*$ . It follows from (18), (19) and (20) that this sequence satisfies the requirements of the theorem.  $\square$

*Remark 5.1.* (a) The case  $u = 0$  in Theorem 5 is known, see, e.g. [9, 10].

(b) The case  $u \neq 0$  is new even for convex lsc functions (recall that such functions are radially accessible at any point  $\bar{x} \in \text{dom } f$  from any  $u$  such that  $\bar{x} + u \in \text{dom } f$ ).

(c) For  $u \neq 0$ , the conclusion of Theorem 5 can be false at points where the function is not radially accessible. Let  $f : \mathbb{R} \rightarrow ]-\infty, +\infty]$  given by

$$f(x) := \begin{cases} 0 & \text{if } x = 0 \text{ or } x = 1/n, \text{ for } n = 1, 2, \dots \\ +\infty & \text{otherwise.} \end{cases}$$

Then,  $f$  is lsc on  $\mathbb{R}$  but not radially accessible at any point  $\bar{x} = 1/n$  from  $u \neq 0$ . We observe that all the points  $x \neq \bar{x}$  close to  $\bar{x}$  are not in  $\text{dom } f$ , hence  $\partial f(x) = \emptyset$ .

(d) For  $u \neq 0$ , we cannot claim in the conclusion of Theorem 5 to find a radially convergent sequence  $(x_n)$  instead of a directionally convergent one. Consider the function  $f : \mathbb{R}^2 \rightarrow ]-\infty, +\infty]$  given, for  $x = (\xi_1, \xi_2)$ , by

$$f(x) := \begin{cases} -\sqrt{\xi_1} & \text{if } (\xi_1, \xi_2) \in \mathbb{R}_+ \times \mathbb{R} \\ +\infty & \text{otherwise.} \end{cases}$$

Then,  $f$  is convex lsc on  $\mathbb{R}^2$  and  $f(0, t) = 0$  for every  $t \in \mathbb{R}$ , so  $f$  is radially continuous at  $\bar{x} = (0, 0)$  in the direction  $u = (0, 1)$ . But, for every  $t \in \mathbb{R}$  we have  $\partial f(0, t) = \emptyset$ , so there is no sequence  $(x_n)$  radially convergent to  $\bar{x}$  from the direction  $u = (0, 1)$  with  $\partial f(x_n) \neq \emptyset$ .

The following lemma comes from our papers [9, 10]. It states an elementary mean value inequality using the radial subderivative.

**Lemma 6** (Basic mean value inequality). *Let  $X$  be a Hausdorff locally convex space,  $f : X \rightarrow ]-\infty, +\infty]$  be lsc,  $\bar{x} \in X$  and  $x \in \text{dom } f$ . Then, for every real number  $\lambda \leq f(\bar{x}) - f(x)$ , there exist  $t_0 \in [0, 1[$  and  $x_0 := x + t_0(\bar{x} - x) \in [x, \bar{x}[$  such that  $f(x_0) \leq f(x) + t_0\lambda$  and*

$$\lambda \leq f^r(x_0; \bar{x} - x).$$

**Theorem 7** (Radial stability of the radial subderivative). *Let  $X$  be a Hausdorff locally convex space,  $f : X \rightarrow ]-\infty, +\infty]$  be lsc,  $\bar{x} \in \text{dom } f$  and  $u \in X$  such that  $f$  is radially accessible at  $\bar{x}$  from  $u$ . Then, there exists a sequence  $\mu_n \searrow 0$  such that  $f(\bar{x} + \mu_n u) \rightarrow f(\bar{x})$  and*

$$f^r(\bar{x}; u) \leq \liminf_{n \rightarrow +\infty} f^r(\bar{x} + \mu_n u; u). \quad (21)$$

*Proof.* If  $f^r(\bar{x}; u) = -\infty$ , there is nothing to prove. Otherwise, it suffices to show that for every  $\lambda < f^r(\bar{x}; u)$ , there exists a sequence  $\mu_n \searrow 0$  such that  $f(\bar{x} + \mu_n u) \rightarrow f(\bar{x})$  and

$$\lambda < \liminf_{n \rightarrow +\infty} f^r(\bar{x} + \mu_n u; u). \quad (22)$$

So, let  $\lambda < f^r(\bar{x}; u)$ . Take  $\lambda', \lambda'_1, \lambda'_2$  such that  $\lambda < \lambda'_1 < \lambda' < \lambda'_2 < f^r(\bar{x}; u)$ . By assumption, there is a sequence  $t_n \searrow 0$  such that  $f(\bar{x} + t_n u) \rightarrow f(\bar{x})$  and by definition of  $f^r(\bar{x}; u)$  there exists  $\tau > 0$  such that

$$\lambda' t < f(\bar{x} + t u) - f(\bar{x}) \quad \text{for all } t \in ]0, \tau].$$

Thus, for large  $n$  we must have  $\lambda' t_n < f(\bar{x} + t_n u) - f(\bar{x})$ . For any such  $n$ , let  $k_n \in \mathbb{N}$  with  $0 < t_{k_n} < t_n$  and

$$\lambda' t_n < f(\bar{x} + t_n u) - f(\bar{x} + t_{k_n} u), \quad (23a)$$

$$\lambda'_1 < \lambda' t_n / (t_n - t_{k_n}) < \lambda'_2. \quad (23b)$$

Applying Lemma 6 to (23a) we obtain  $t_0 \in [0, 1[$  and  $x_0 := \bar{x} + t_{k_n} u + t_0(t_n - t_{k_n})u = \bar{x} + \mu_n u$  with  $\mu_n := t_{k_n} + t_0(t_n - t_{k_n}) \in [t_{k_n}, t_n[$  such that

$$f(\bar{x} + \mu_n u) \leq f(\bar{x} + t_{k_n} u) + t_0 \lambda' t_n, \quad (24a)$$

$$\lambda' t_n < f^r(\bar{x} + \mu_n u; (t_n - t_{k_n})u). \quad (24b)$$

Using (23b) and the definition of  $\mu_n$ , from (24a)–(24b) we derive

$$f(\bar{x} + \mu_n u) \leq f(\bar{x} + t_{k_n} u) + \lambda'_2(\mu_n - t_{k_n}), \quad (25a)$$

$$\lambda'_1 < f^r(\bar{x} + \mu_n u; u). \quad (25b)$$

Then, letting  $n \rightarrow +\infty$  we obtain

$$\mu_n \searrow 0, \quad \limsup_{n \rightarrow +\infty} f(\bar{x} + \mu_n u) \leq f(\bar{x}), \quad \lambda < \lambda'_1 \leq \liminf_{n \rightarrow +\infty} f^r(\bar{x} + \mu_n u; u).$$

This completes the proof since we also have  $f(\bar{x}) \leq \liminf_{n \rightarrow +\infty} f(\bar{x} + \mu_n u)$  by the lower semicontinuity of  $f$  at  $\bar{x}$ .  $\square$

*Remark 7.1.* (a) If the assumption that  $f$  is radially accessible at  $\bar{x}$  from  $u$  is strengthened to  $f$  is radially upper semicontinuous at  $\bar{x}$  from  $u$ , this will not strengthen the conclusion to  $f^r(\cdot; u)$  is radially lower semicontinuous at  $\bar{x}$  from  $u$ : the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$f(x) := \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous and differentiable on  $\mathbb{R}$ ; for  $x \neq 0$ , we have  $f^r(x; 1) = 2x \sin(1/x) - \cos(1/x)$  and  $f^r(0; 1) = 0$ , so  $f^r(0; 1) \not\leq \liminf_{t \searrow 0} f^r(0+t; 1) = -1$ ; the function  $f^r(\cdot; u)$  is not radially lower semicontinuous at 0 from  $u = 1$ ; but (21) is satisfied with  $\mu_n = 1/(2n+1)\pi$ .

(b) In (21) we may have  $f^r(\bar{x}; u) = \infty$ . For instance, for the continuous  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) := \sqrt{|x|}$ , one has, for any  $u \neq 0$ ,

$$f^r(0; u) = \infty \leq \limsup_{t \searrow 0} f^r(0 + tu; u) = \limsup_{t \searrow 0} \frac{|u|}{2\sqrt{t|u|}} \cdot u = \infty.$$

(c) The assumption that  $f$  is radially accessible at  $\bar{x}$  from  $u$  cannot be omitted: Example 4 above shows a lsc  $f : \mathbb{R} \rightarrow \mathbb{R}$  which is not radially accessible at 0 from  $u = 1$ , and we observed that  $f^r(0; 1) = +\infty > \limsup_{t \searrow 0} f^r(0+t; 1) = 0$ .

It follows from Theorem 7 that for  $f$  radially accessible at  $\bar{x}$  from  $u$ , the radial subderivative  $f^r(\bar{x}; u)$  is a lower bound for the expressions in (2) with  $v = u$ :

**Theorem 8** (Directional stability of the radial subderivative). *Let  $X$  be a Banach space,  $f : X \rightarrow ]-\infty, +\infty]$  be lsc,  $\bar{x} \in \text{dom } f$  and  $u \in X$  such that  $f$  is radially accessible at  $\bar{x}$  from  $u$ . Then,*

$$f^r(\bar{x}; u) \leq \inf_{\alpha \geq 0} \limsup_{x \rightarrow_u \bar{x}} f^r(x; u + \alpha(\bar{x} - x)). \quad (26)$$

*Proof.* By Theorem 7, there is a sequence  $x_n := \bar{x} + \mu_n u$  with  $\mu_n \searrow 0$  such that

$$f^r(\bar{x}; u) \leq \liminf_{n \rightarrow +\infty} f^r(x_n; u). \quad (27)$$

Since  $u + \alpha(\bar{x} - x_n) = (1 - \alpha\mu_n)u$ , it follows that, for any  $\alpha \geq 0$ ,

$$\liminf_{n \rightarrow +\infty} f^r(x_n; u + \alpha(\bar{x} - x_n)) = \liminf_{n \rightarrow +\infty} (1 - \alpha\mu_n) f^r(x_n; u) = \liminf_{n \rightarrow +\infty} f^r(x_n; u). \quad (28)$$

Since  $x_n \rightarrow_u \bar{x}$ , we derive from (27) and (28) that, for every  $\alpha \geq 0$ ,

$$f^r(\bar{x}; u) \leq \liminf_{n \rightarrow +\infty} f^r(x_n; u + \alpha(\bar{x} - x_n)) \leq \limsup_{x \rightarrow_u \bar{x}} f^r(x; u + \alpha(\bar{x} - x)). \quad \square$$

## 6 Refined link between radial subderivative and subdifferentials

Plugging the formula (2b) of Theorem 3 into the inequality (26) of Theorem 8, we immediately obtain an inequality linking radial subderivative and subdifferentials:

$$f^r(\bar{x}; u) \leq \inf_{\alpha \geq 0} \limsup_{x \rightarrow_v \bar{x}} \langle \partial f(x), u + \alpha(\bar{x} - x) \rangle. \quad (29)$$

For the sake of completeness, we provide a direct proof of this inequality. In fact, we shall establish an inequality more accurate than (29), in the same vein as Theorem 2 and Theorem 5.

**Theorem 9** (Refined link between radial subderivative and subdifferentials). *Let  $X$  be a Banach space,  $f : X \rightarrow ]-\infty, +\infty]$  be lsc,  $\bar{x} \in \text{dom } f$  and  $u \in X$  such that  $f$  is radially accessible at  $\bar{x}$  from  $u$ . Then, there is a sequence  $((x_n, x_n^*)) \subset \partial f$  such that  $x_n \rightarrow_u \bar{x}$ ,  $f(x_n) \rightarrow f(\bar{x})$  and*

$$f^r(\bar{x}; u) \leq \liminf_n \langle x_n^*, u + \alpha(\bar{x} - x_n) \rangle, \quad \forall \alpha \geq 0. \quad (30)$$

*Proof.* The pattern of the proof is similar to that of [10, Theorem 2.1] but the argument has to be refined in order to obtain a directionally convergent sequence.

*First step.* If  $u = 0$  or if  $f^r(\bar{x}; u) = -\infty$ , the result follows from Theorem 5. Otherwise, assume  $u \neq 0$ , let  $\gamma < f^r(\bar{x}; u)$  and let  $\varepsilon > 0$ . We claim that for each  $n \in \mathbb{N}$  sufficiently large, there exists  $(x_n, x_n^*) \in \hat{\partial} f$  such that

$$x_n \in D(\bar{x}, u, \varepsilon), \quad f(x_n) < f(\bar{x}) + \varepsilon, \quad (31a)$$

$$\langle x_n^*, u + \alpha(\bar{x} - x_n) \rangle > \gamma - (\alpha + 1)\varepsilon, \quad \forall \alpha \geq 0. \quad (31b)$$

Take  $t_0 \in ]0, 1[$  such that

$$f(\bar{x}) \leq f(\bar{x} + tu) - \gamma t, \quad \text{for all } t \in [0, t_0].$$

Let  $z^* \in X^*$  such that  $\langle z^*, u \rangle = -\gamma$ , set  $g := f + z^*$  and  $K := [\bar{x}, \bar{x} + t_0 u]$ . Let also  $\delta > 0$  such that  $g$  is bounded below on  $B_\delta(K)$ .

By Theorem 7, there exists a sequence  $\mu_n \searrow 0$  such that

$$|f(\bar{x} + \mu_n u) - f(\bar{x})| < 1/n, \quad (32a)$$

$$\gamma < f^r(\bar{x} + \mu_n u; u). \quad (32b)$$

We may assume  $0 < \mu_n < \min\{t_0, \sqrt{\delta}\}$ . By (32b), there exists  $t_n \in ]0, t_0 - \mu_n]$  such that

$$f(\bar{x} + \mu_n u) \leq f(\bar{x} + \mu_n u + t_n u) - \gamma t_n, \quad \forall t \in [0, t_n]. \quad (33)$$

Let  $K_n := [\bar{x} + \mu_n u, \bar{x} + (\mu_n + t_n)u] \subset K$ . Then, (33) can be rewritten as

$$g(\bar{x} + \mu_n u) \leq g(x), \quad \forall x \in K_n. \quad (34)$$

Take  $r > 0$  such that

$$g(\bar{x} + \mu_n u) < \inf_{B_r(K_n)} g + \mu_n^3 t_n, \quad (35)$$

and, observing that both  $\inf_{B_r(K_n)} g$  and  $\inf_{B_\delta(K_n)} g$  are finite, choose  $\alpha_n > 0$  such that

$$\inf_{B_r(K_n)} g \leq \inf_{B_\delta(K_n)} g + \alpha_n r^2.$$

Then

$$\inf_{B_r(K_n)} g \leq (g + \alpha_n d_{K_n}^2)(x), \quad \forall x \in B_\delta(K_n),$$

and therefore, by (35),

$$g(\bar{x} + \mu_n u) \leq (g + \alpha_n d_{K_n}^2)(x) + \mu_n^3 t_n, \quad \forall x \in B_\delta(K_n). \quad (36)$$

Now, apply Ekeland's variational principle to the function  $g + \alpha_n d_{K_n}^2$  on the set  $B_\delta(K_n)$  at point  $\bar{x} + \mu_n u \in K_n$  with  $\varepsilon = \mu_n^3 t_n$  and  $\lambda = \mu_n^2 t_n$ . Observe that the ball  $B_\lambda(\bar{x} + \mu_n u)$  is contained in  $B_\delta(K_n)$  since for every  $x \in B_\lambda(\bar{x} + \mu_n u)$ , we have  $d_{K_n}(x) \leq \|x - (\bar{x} + \mu_n u)\| \leq \lambda = \mu_n^2 t_n < \delta$ . We then obtain a point  $x_n \in X$  satisfying

$$\|x_n - (\bar{x} + \mu_n u)\| < \mu_n^2 t_n, \quad g(x_n) + \alpha_n d_{K_n}^2(x_n) \leq g(\bar{x} + \mu_n u) \quad (37a)$$

$$y \mapsto f(y) + \langle z^*, y \rangle + \alpha_n d_{K_n}^2(y) + \mu_n \|y - x_n\| \text{ admits a local minimum at } x_n. \quad (37b)$$

It follows from the first half of (37a) that

$$x_n - \bar{x} \in B(\mu_n u, \mu_n^2 t_n) = \mu_n B(u, \mu_n t_n),$$

showing that  $x_n \in D(\bar{x}, u, \varepsilon)$  for  $n$  sufficiently large. On the other hand, the second half of (37a) and (32a) entail

$$f(x_n) \leq f(\bar{x} + \mu_n u) + \|z^*\| \|\bar{x} + \mu_n u - x_n\| \leq f(\bar{x}) + \|z^*\| \mu_n^2 t_n + 1/n,$$

showing that  $f(x_n) < f(\bar{x}) + \varepsilon$  for  $n$  sufficiently large.

In view of (37b), we may apply the Separation Principle at point  $x_n$  with the convex Lipschitz function  $\varphi : y \mapsto \langle z^*, y \rangle + \alpha_n d_{K_n}^2(y) + \mu_n \|y - x_n\|$  to obtain points  $x_n^* \in \hat{\partial} f(x_n)$ ,  $\zeta_n^* \in \partial d_{K_n}^2(x_n)$  and  $\beta_n^* \in B^*$  with

$$0 = x_n^* + z^* + \alpha_n \zeta_n^* + \mu_n \beta_n^*. \quad (38)$$

We claim that the pair  $(x_n, x_n^*) \in \hat{\partial} f$  satisfies (31b) for large  $n \in \mathbb{N}$ . Assume it can be shown that for all  $\alpha \geq 0$  and for large  $n \in \mathbb{N}$ ,

$$\langle \zeta_n^*, u + \alpha(\bar{x} - x_n) \rangle \leq 0. \quad (39)$$

Then, it follows that for all  $\alpha \geq 0$  and for large  $n \in \mathbb{N}$ ,

$$\begin{aligned} \langle x_n^*, u + \alpha(\bar{x} - x_n) \rangle &= \langle -z^*, u + \alpha(\bar{x} - x_n) \rangle - \alpha_n \langle \zeta_n^*, u + \alpha(\bar{x} - x_n) \rangle - 2\mu_n \langle \beta_n^*, u + \alpha(\bar{x} - x_n) \rangle \\ &\geq \gamma - \alpha \|z^*\| \|\bar{x} - x_n\| - 2\mu_n \|u + \alpha(\bar{x} - x_n)\|, \end{aligned}$$

which implies that  $\langle x_n^*, u + \alpha(\bar{x} - x_n) \rangle > \gamma - (\alpha + 1)\varepsilon$  for  $n$  sufficiently large, as claimed.

*Second step.* To complete the proof of (31b) it remains to prove (39). We first consider the case  $\alpha = 0$ , that is, we show

$$\langle \zeta_n^*, u \rangle \leq 0, \quad \forall n \in \mathbb{N}. \quad (40)$$

Let  $P_{K_n} x_n \in K_n$  be any point such that  $\|x_n - P_{K_n} x_n\| = d_{K_n}(x_n)$ . We have

$$\|\bar{x} + \mu_n u - P_{K_n} x_n\| \leq \|\bar{x} + \mu_n u - x_n\| + \|x_n - P_{K_n} x_n\| < 2\mu_n^2 t_n.$$

So,  $P_{K_n}x_n = \bar{x} + \mu_n u + \tau_n u$  with  $\tau_n \|u\| < 2\mu_n^2 t_n$ . Hence,  $t_n - \tau_n > 0$  for large  $n$ , and

$$(t_n - \tau_n)u = \bar{x} + (\mu_n + t_n)u - P_{K_n}x_n.$$

Notice that  $\zeta_n^* = 2d_{K_n}(x_n)\xi_n^*$  where  $\xi_n^* \in \partial d_{K_n}(x_n)$ . Then:

$$\begin{aligned} (t_n - \tau_n)\langle \zeta_n^*, u \rangle &= \langle \zeta_n^*, \bar{x} + (\mu_n + t_n)u - P_{K_n}x_n \rangle \\ &= \langle \zeta_n^*, \bar{x} + (\mu_n + t_n)u - x_n \rangle + \langle \zeta_n^*, x_n - P_{K_n}x_n \rangle \\ &= 2d_{K_n}(x_n) (\langle \xi_n^*, \bar{x} + (\mu_n + t_n)u - x_n \rangle + \langle \xi_n^*, x_n - P_{K_n}x_n \rangle) \\ &\leq 2d_{K_n}(x_n)(-d_{K_n}(x_n) + \|x_n - P_{K_n}x_n\|) = 0. \end{aligned}$$

This proves (40).

Now consider the case  $\alpha > 0$ . Write  $\alpha = 1/t$ . We must show that for large  $n \in \mathbb{N}$ ,

$$\langle \zeta_n^*, u + \alpha(\bar{x} - x_n) \rangle = (1/t)\langle \zeta_n^*, \bar{x} + tu - x_n \rangle \leq 0. \quad (41)$$

But, for  $n$  so large that  $\mu_n < t$ , we have

$$\begin{aligned} \langle \zeta_n^*, \bar{x} + tu - x_n \rangle &= \langle \zeta_n^*, \bar{x} + \mu_n u - x_n \rangle + \langle \zeta_n^*, (t - \mu_n)u \rangle \\ &\leq d_{K_n}^2(\bar{x} + \mu_n u) - d_{K_n}^2(x_n) + (t - \mu_n)\langle \zeta_n^*, u \rangle \\ &\leq -d_{K_n}^2(x_n) \quad \text{by (40)} \\ &\leq 0. \end{aligned}$$

This proves (41). Hence, (39) holds and so also (31b), as we have observed.

*Third step.* Every pair  $(x_n, x_n^*)$  in  $\widehat{\partial}f$  is close to a pair  $(\bar{x}_n, \bar{x}_n^*)$  in  $\partial f$  in such a way that the sequence  $((x_n, x_n^*))_n$  satisfying (31a)–(31b) for large  $n \in \mathbb{N}$  can actually be assumed to lie in  $\partial f$  (proceed as in Theorem 5).

*Fourth step.* The theorem is derived from (31a)–(31b) as follows. Let  $(\gamma_k)_k$  be an increasing sequence of real numbers such that  $\gamma_k \nearrow f^r(\bar{x}; d)$ . We have proved that, for each  $k \in \mathbb{N}$ , there are a sequence  $((x_{n,k}, x_{n,k}^*))_n \subset \partial f$  and an integer  $N_k \in \mathbb{N}$  satisfying for every  $n \geq N_k$ :

$$x_{n,k} \in D(\bar{x}, u, 1/k), \quad f(x_{n,k}) < f(\bar{x}) + 1/k, \quad (42a)$$

$$\langle x_{n,k}^*, u + \alpha(\bar{x} - x_{n,k}) \rangle > \gamma_k - (\alpha + 1)/k, \quad \forall \alpha \geq 0. \quad (42b)$$

Clearly, we may assume  $N_{k+1} > N_k$ . Then, it is immediate from (42a)–(42b) that the diagonal sequence defined, for  $k \in \mathbb{N}$ , by  $(x_k, x_k^*) := (x_{N_k, k}, x_{N_k, k}^*)$  satisfies the assertions of the theorem.  $\square$

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